

EINSTEIN'S EQUATIONS WITH ASYMPTOTICALLY STABLE CONSTRAINT PROPAGATION *

OTHMAR BRODBECK [†]

Center for Gravitational Physics and Geometry
Department of Physics, The Pennsylvania State University
University Park, PA 16802, USA

and

Institute for Theoretical Physics, The University of Zurich
Winterthurerstrasse 190, 8057 Zurich, Switzerland

SIMONETTA FRITTELLI [‡]

Duquesne University
Pittsburgh, Pennsylvania, 15282, USA

PETER HÜBNER [§]

Albert-Einstein-Institut
Max-Planck-Institut für Gravitationsphysik
Schlaatzweg 1, 14473 Potsdam, Germany

OSCAR A. REULA [¶]

FaMAF, Medina Allende y Haya de la Torre,
Ciudad Universitaria, 5000 Córdoba, Argentina

and

Albert-Einstein-Institut
Max-Planck-Institut für Gravitationsphysik
Schlaatzweg 1, 14473 Potsdam, Germany

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Abstract

We introduce a proposal to modify Einstein's equations by embedding them in a larger symmetric hyperbolic system. The additional dynamical variables of the modified system are essentially first integrals of the original constraints. The extended system of equations reproduces the usual dynamics on the constraint surface of general relativity, and therefore naturally includes the solutions to Einstein gravity. The main feature of this extended system is that, at least for a linearized version of it, the constraint surface is an attractor of the time evolution. This feature suggests that this system may be a useful alternative to Einstein's equations when obtaining numerical solutions to full, non-linear gravity.

1 Introduction

Over the past decade, computer power has increased to the point that simulations of two- and even three-dimensional general relativity are now feasible. These simulations, which assume little or no symmetry of their generic field configurations, at first seemed to represent straightforward generalizations of simpler one-dimensional calculations. However, attempts to perform the higher dimensional simulations have revealed a variety of unexpected features which limit accurate simulations to a rather short time interval. One such feature, which is believed to be a major source of numerical error, is that numerical time evolution generates a rapidly growing violation of the constraint equations. In this paper, we propose a system of dynamical equations wherein the evolution naturally remains close to the constraint surface. Although the most obvious application of this approach is to numerical simulations, it may prove useful in other branches of general relativity as well.

As is well known analytically, the time evolution predicted by the exact Einstein equations is such that the constraint equations are satisfied on each time slice when they are satisfied by the initial data. Geometrically, the evolution vector field is tangential to the constraint submanifold, implying that solutions to the complete set of equations are insensitive to properties of the

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†email: brodbeck@phys.psu.edu

‡email: simo@mayu.physics.duq.edu

§email: pth@aei-potsdam.mpg.de

¶Member of CONICET, email: reula@aei-potsdam.mpg.de

evolution field in the vicinity of the constraint surface. In discrete approximations, on the other hand, the notion of tangency is approximate, as is that of the constraint surface itself. As a consequence, the numerical evolution becomes sensitive to possible instabilities of the constraint submanifold and numerical solutions are, in general, carried away from it exponentially fast with time. Even in case one were able to construct a code whose discretized vector field were exactly tangent to a discretised version of the constraint submanifold, the same problem would be likely to arise, as numerical errors on the initial data would prevent a start of the time integration exactly on the constraint submanifold.

As demonstrated in [Cho91], evolution schemes can be constructed in such a way that the violation of the constraints has the same convergence order as the scheme itself. This property, which is in the meantime a standard requirement for evolution schemes, implies that the choice of an appropriately fine grid is sufficient to satisfy the constraint equations at any given time with arbitrary accuracy. However, since the violation of the constraints grows very quickly with time, the utility of grid refinements to reduce constraint violations is very limited, especially in two- and three-dimensional calculations.

In the so-called constrained evolution schemes one, attempts to solve this problem by isolating two sets of variables in Einstein's equations. One uses evolution equations to evolve one set and determines the variables of the other set by solving constraint equations on each time slice. This method has frequently been used in one-dimensional simulations where, on the one hand, it is easy to split the variables into dynamical and longitudinal ones, and where, on the other hand, the constraint equations are ordinary differential equations along a space-like direction. However, in two- and three-dimensional simulations with space-like hypersurfaces as time surfaces, the elliptic character of the constraint equations makes it expensive in computer time to solve the constraint equations on each time slice.¹ Furthermore, this approach does not guarantee that the complete set of Einstein's equations is solved. Since only a subset of the variables is determined by evolution equations, some of these equations remain unused. The problem is, therefore, shifted to the preservation of the unused evolution equations, which, as shown in [Det87], is a problem of similar nature.

In order to guarantee a good approximation to the complete set of field

¹Not to mention the problems arising in the treatment of the grid boundaries.

equations, it is, therefore, necessary to analyze the behavior of the evolution vector field in a whole neighborhood of the constraint submanifold. Away from the constraint submanifold the evolution field is not uniquely determined as field configurations violating the constraint equations are physically not relevant. Hence, the evolution vector field can be modified in an arbitrary way, as long as its values on the constraint submanifold remain unchanged, and as long as the modified field continues to be strongly hyperbolic, so that the Cauchy problem is well posed in a whole neighborhood of the constraint surface.

Of particular interest are modified equations for which the constraint submanifold is asymptotically stable, because for equations with this feature, sufficiently accurate codes are expected to generate solutions which remain close to the constraint surface, and which, therefore, would represent improved approximations to Einstein's equations.

Modifications of the evolution vector field have previously been studied. However, in general these preserve the time reversal symmetry of Einstein's equations, which implies that modifications of this type cannot have the desired properties. Time reversal symmetry implies that if the evolution field is such that a solution to some initial data in a neighborhood of the constraint submanifold approaches the constraint submanifold during time evolution, then the solution to the time reversed initial data will asymptotically be carried away from this submanifold. Thus, without a modification of Einstein's equations which breaks the time reversal symmetry, the best one can expect to achieve is a set of equations for which the constraint submanifold is stable, but not asymptotically stable. However, stability of the undiscretized equations is not sufficient for numerical simulations, since spurious solutions to the discretized equations can grow very rapidly even for stable systems. To eliminate the impact of such solutions, it is, therefore, necessary that the constraint manifold is an attractor for the time evolution.

The above-mentioned modifications of Einstein's equations are, in general, obtained by including dynamical quantities which are proportional to the constraint expressions. An alternative argument showing that extensions of this type cannot lead to an asymptotically stable constraint surface is the following: Since the constraint equations are of the same order as the evolution equations, their inclusion affects mainly the principal part of the evolution equations, whence the freedom remaining after requiring that these terms do not destroy strong hyperbolicity is very limited. Thus, such extensions ensure the well posedness of the problem, but not the asymptotic

stability. This can only be obtained either via modifications of the lower order terms or the addition of higher (second) order terms, that is, by including damping or diffusion terms.

In the next section, we propose a modification of Einstein's equations which includes new dynamical terms proportional to certain first integrals of the constraint expressions, rather than to the constraints themselves. The dissipation, that is the time asymmetry is not of the diffusive type,² and is built into the definition of these integrals.

We show that the Cauchy problem for the resulting new system, which we call the λ -**system**,³ is locally well posed. We also prove that if the constraints are initially satisfied, and if their first integrals initially vanish, then the λ -system provides solutions to Einstein's equations. Moreover, for initial data sets for the λ -system, which are sufficiently close to the constraint submanifold and sufficiently close to zero, respectively, we suspect that the solutions asymptotically tend to solutions to Einstein's equations.

In section 3, we give support to our expectation by proving that the linearized extended system is asymptotically stable, thus showing that in the linearized case, the constraint submanifold is indeed an attractor for the λ -system. In section 4, we discuss further expectations in connection with our proposal.

2 The λ -system

In this section we spell out our proposal for a modification of Einstein's equations with an asymptotically stable constraint submanifold. For definiteness, we choose the symmetric hyperbolic system introduced by Frittelli and Reula in [FR94], which corresponds to the parameters $\alpha = \beta = -1$ in [FR96]. Although the full equations (with the non-principal part terms added) are given in [Ste98], we repeat them for completeness.

In the version of Einstein's vacuum equations chosen, the system is given by the following set of dynamical equations (where Latin indices run from 1 to 3):

$$\dot{h}^{ij} = N^n h^{ij}_{,n} + Q\sqrt{h} \left(2P^{ij} - Ph^{ij} \right) - 2h^{n(i} N^{j)},_n, \quad (1)$$

²One could also introduce diffusive dissipation, but this would significantly reduce the allowed maximal time step in explicit discretisation schemes.

³The name is a remnant of the way the system was originally guessed by Brodbeck and Hübner, namely by a formal application of Lagrangian multiplier techniques.

$$\begin{aligned}
\dot{M}^{ij}_k &= N^n M^{ij}_{k,n} + Q\sqrt{h} \left(P^{ij}_{,k} - 2\delta_k^{(i} P^{j)n}_{,n} \right) \\
&\quad + Q\sqrt{h} \left(\frac{3}{2} P^{ij} M_k - P M^{ij}_k + Q^{-1} P^{ij} Q_{,k} \right. \\
&\quad \left. - 2\delta_k^{(i} \left[h^{j)q} h_{mr} h_{ns} P^{mn} M^{rs}_q - 2M^{j)p}_n P^{mn} h_{pm} \right. \right. \\
&\quad \left. \left. + \frac{3}{2} P^{j)n} M_n - \frac{1}{2} h^{j)n} P M_n \right] \right) \\
&\quad + h^{ij} N^n_{,nk} - h^{n(i} N^{j)}_{,nk} - 2N^{(i}_n M^{j)n}_k + N^m_{,k} M^{ij}_m, \quad (2) \\
\dot{P}^{ij} &= N^n P^{ij}_{,n} + Q\sqrt{h} \left(h^{mn} M^{ij}_{m,n} - 2h^{n(i} M^{j)k}_{,k,n} \right) \\
&\quad + Q\sqrt{h} \left(4h_{np} h^{m(i} M^{j)n}_k M^{kp}_m - h^{ik} h^{jn} h_{rp} h_{sq} M^{rs}_k M^{pq}_n \right. \\
&\quad + \frac{1}{2} h^{ik} h^{jn} M_k M_n + 2M^{nk}_k M^{ij}_n - 2M^{ik}_n M^{jn}_k \\
&\quad - 2h_{mn} h^{kp} M^{im}_k M^{jn}_p - 2h^{n(i} M^{j)k}_k M_n + M^{ij}_n h^{nk} M_k \\
&\quad - Q^{-1} \left[h^{ik} h^{jn} Q_{,kn} + 2M^{k(i}_n h^{j)n} Q_{,k} - M^{ij}_m h^{mk} Q_{,k} \right. \\
&\quad \left. - h^{ij} \left(h^{kn} Q_{,kn} + 2M^{km}_m Q_{,k} \right) \right] + 2P^{ik} h_{kn} P^{nj} - \frac{3}{2} P P^{ij} \\
&\quad \left. + h^{ij} \left(\frac{1}{2} P^2 - h_{mr} h_{ns} P^{mn} P^{rs} \right) \right) - 2P^{k(i} N^{j)}_{,k}. \quad (3)
\end{aligned}$$

Here, h^{ij} is the inverse intrinsic metric of the spacelike hypersurfaces Σ_t , $P^{ij} := k^{ij} - h^{ij}k$ denotes a linear combination of the extrinsic curvature k^{ij} of the slice and its trace k , and $M^{ij}_k := \frac{1}{2} (h^{ij}_{,k} - h^{ij} h_{rs} h^{rs}_{,k})$ represents a linear combination of spatial derivatives of the inverse intrinsic metric. The functions P and M_k are abbreviations for $h_{ij} P^{ij}$ and $h_{ij} M^{ij}_k$, respectively, and Q and N^i are arbitrary given functions fixing the gauge degrees of freedom.

This evolution system is symmetric hyperbolic with respect to the inner product

$$\begin{aligned}
\langle h^{ij}_1, P^{ij}_1, M^{ij}_k \mid H_e \mid h^{ij}_2, P^{ij}_2, M^{ij}_k \rangle &:= \\
\int_{\Sigma_t} \left\{ e_{im} e_{jn} \bar{h}^{ij}_1 h^{mn}_2 + e_{im} e_{jn} \bar{P}^{ij}_1 P^{mn}_2 + e_{im} e_{jn} e^{kl} \bar{M}^{ij}_k M^{mn}_l \right\} d\Sigma, \quad (4)
\end{aligned}$$

where e_{ij} denotes an Euclidean flat metric on the hypersurface Σ_t . It is completed by the following set of constraints equations:

$$\mathcal{C} = 0, \quad \mathcal{C}^i = 0, \quad \mathcal{C}^{ij}_k = 0, \quad (5)$$

where

$$\begin{aligned} \mathcal{C} := & -M^{kn}_{n,k} + h_{pq}M^{pk}_nM^{qn}_k - M^{kq}_qM_k + \frac{1}{4}h^{kn}M_kM_n \\ & - \frac{1}{2}h_{mn}h_{rs}h^{pq}M^{mr}_pM^{ns}_q - \frac{1}{2}h_{mn}h_{rs}P^{mr}P^{ns} + \frac{1}{4}P^2, \end{aligned} \quad (6)$$

$$\begin{aligned} \mathcal{C}^i := & P^{ik}_{,k} - 2h_{mn}M^{im}_kP^{nk} - \frac{1}{2}h^{ik}PM_k \\ & + h_{mn}h_{pq}h^{ik}M^{mp}_kP^{nq} + \frac{3}{2}P^{ik}M_k, \end{aligned} \quad (7)$$

$$\mathcal{C}^{ij}_k := 2M^{ij}_k - h^{ij}h_{pq}M^{pq}_k - h^{ij}_{,k}. \quad (8)$$

The first two constraints are the scalar and the vector constraint of Einstein's equations, that is the time-time and time-space components of the Einstein tensor for a given 3+1 decomposition of space-time. The third is the statement that the tensor M^{ij}_k is a linear combination of spatial derivatives of the 3-metric.

To solve the initial value problem of general relativity in this approach, one prescribes an initial data set $(h_0^{ij}, P_0^{ij}, M_0^{ij}_k)$ at $t = 0$ which satisfies the constraints equations and subsequently solves the above evolution equations. Symmetric hyperbolicity of the evolution system implies that a unique local solution does exist.

By taking a time derivative of equations (6–8) and using (1–3) to eliminate time derivatives in favor of spatial derivatives, the following evolution equations for the constraints are obtained:

$$\dot{\mathcal{C}} = N^n\mathcal{C}_{,n} + 3Q\sqrt{h}\mathcal{C}^k_{,k} + \dots, \quad (9)$$

$$\begin{aligned} \dot{\mathcal{C}}^i = & N^n\mathcal{C}^i_{,n} + Q\sqrt{h}\left(h^{ik}\mathcal{C}_{,k} + h^{rs}\mathcal{C}^{ik}_{[r,k]s} + h^{is}h^{kl}h_{mn}\mathcal{C}^{mn}_{[s,k]l}\right) \\ & + \dots, \end{aligned} \quad (10)$$

$$\dot{\mathcal{C}}^{ij}_k = N^n\mathcal{C}^{ij}_{k,n} - 2Q\sqrt{h}\left(2\delta_k^{(i}\mathcal{C}^{j)} - h^{ij}h_{kl}\mathcal{C}^l\right) + \dots, \quad (11)$$

where “...” represent undifferentiated terms which are linear in the constraint quantities and at least linear in the variables P^{ij} and M^{ij}_k .

Since equation (10) is of second order in spatial derivatives, we introduce a further constraint by⁴

$$C^{ij}_{kl} := 2M^{ij}_{[k,l]} + 2M^{ij}_{[k}M_{l]}. \quad (12)$$

⁴One could also consider the constraint $C^i_j := C^{ik}_{jk}$, which still makes the constraint system symmetric hyperbolic and produces a smaller number of extra fields.

By taking a time derivative of (12), we obtain

$$\dot{\mathcal{C}}^{ij}_{kl} = N^n \mathcal{C}^{ij}_{kl,n} - 2Q\sqrt{h} \left(\delta_k^{(i} \mathcal{C}^{j)},_{,l} - \delta_l^{(i} \mathcal{C}^{j)},_{,k} \right) + \dots, \quad (13)$$

and by plugging (12) into (10), we see that the evolution equation for \mathcal{C}^i can be rewritten as

$$\dot{\mathcal{C}}^i = N^n \mathcal{C}^i_{,n} + Q\sqrt{h} \left(h^{ik} \mathcal{C}_{,k} + h^{rs} \mathcal{C}^{ik}_{rk,s} \right) + \dots. \quad (14)$$

The constraint quantities \mathcal{C} , \mathcal{C}^i , \mathcal{C}^{ij}_k , and \mathcal{C}^{ij}_{kl} thus propagate according to the first-order system of equations consisting of (9), (11), and (13–14), which is symmetric hyperbolic with respect to the following inner product:

$$\begin{aligned} \langle C_1, C_1^i, C_1^{ij}_k, C_1^{ij}_{kl} \mid H_C \mid C_2, C_2^i, C_2^{ij}_k, C_2^{ij}_{kl} \rangle := \\ \int_{\Sigma_t} \left\{ \frac{1}{3} \bar{C}_1 C_2 + e_{ij} \bar{C}_1^i C_2^j + e_{ij} e_{kl} e^{rs} \bar{C}_1^{ik}{}_r C_2^{jl}{}_s \right. \\ \left. + \frac{1}{4} e_{im} e_{jn} e^{kp} e^{lq} \bar{C}_1^{ij}{}_{kl} C_2^{mn}{}_{pq} \right\} d\Sigma. \end{aligned} \quad (15)$$

Uniqueness of solutions to this system implies that if the constraints are initially satisfied, then the exact evolution equations preserve them. When, as in numerical simulations, the constraint variables initially are not precisely zero, then the corresponding solution is, in general, carried away from the constraint surface during time evolution. However, since the evolution equations for the constraint variables are symmetric hyperbolic, the violation of the constraints becomes smaller when the constraints initially are satisfied with better accuracy.

In order to obtain a system with an asymptotically stable constraint submanifold, we propose a modification of Einstein's equations, which is inspired by the behavior of dissipative systems, where a transient eventually is dissipated away as the system settles down. We extend the set of dynamical variables by considering the following “time integrals” of the constraint variables:

$$\dot{\lambda} = \alpha_0 \mathcal{C} - \beta_0 \lambda, \quad (16)$$

$$\dot{\lambda}^i = \alpha_1 \mathcal{C}^i - \beta_1 \lambda^i, \quad (17)$$

$$\dot{\lambda}^{ij}_k = \alpha_3 \mathcal{C}^{ij}_k - \beta_3 \lambda^{ij}_k, \quad (18)$$

$$\dot{\lambda}^{ij}_{kl} = \alpha_4 \mathcal{C}^{ij}_{kl} - \beta_4 \lambda^{ij}_{kl}, \quad (19)$$

where the tensor-valued λ -variables are assumed to have the same symmetries as the corresponding C -variables, and where $\alpha_i \neq 0$ and $\beta_i > 0$ are constants.

The equations (16–19) represent evolution equations for the λ -variables which in terms of the fundamental variables $(h^{ij}, P^{ij}, M^{ij}_k)$ are given by

$$\begin{aligned} \dot{\lambda} &= \alpha_0 \left(-M^{kn}_{n,k} + h_{pq} M^{pk}_n M^{qn}_k - M^{kq}_q M_k + \frac{1}{4} h^{kn} M_k M_n \right) \\ &\quad - \beta_0 \lambda, \end{aligned} \quad (20)$$

$$\dot{\lambda}^i = \alpha_1 \left(P^{ik}_{,k} - 2h_{mn} M^{im}_k P^{nk} - \frac{1}{2} h^{ik} P M_k \right) - \beta_1 \lambda^i, \quad (21)$$

$$\dot{\lambda}^{ij}_k = \alpha_3 \left(2M^{ij}_k - h^{ij} h_{rs} M^{rs}_k - h^{ij}_{,k} \right) - \beta_3 \lambda^{ij}_k, \quad (22)$$

$$\dot{\lambda}^{ij}_{kl} = \alpha_4 \left(2M^{ij}_{[k,l]} + 2M^{ij}_{[k} M_{l]} \right) - \beta_4 \lambda^{ij}_{kl}. \quad (23)$$

In the present form, the combined system (1–3,20–23) is not symmetric hyperbolic, since the equations (20–23) involve spatial derivatives of the variables $(h^{ij}, P^{ij}, M^{ij}_k)$, whereas the equations (1–3) do not contain λ -variables at all. However, by adding terms containing first derivatives of the λ -variables it is possible to bring the system (1–3,20–23) into a symmetric hyperbolic form,

$$\dot{h}^{ij} = \alpha_3 h^{mn} \lambda^{ij}_{m,n} + N^n h^{ij}_{,n} + \text{source terms}, \quad (24)$$

$$\begin{aligned} \dot{M}^{ij}_k &= 2\alpha_4 h^{lm} \lambda^{ij}_{kl,m} - \alpha_0 \delta_k^{(i} h^{j)l} \lambda_{,l} \\ &\quad + N^n M^{ij}_{k,n} + Q\sqrt{h} \left(P^{ij}_{,k} - 2\delta_k^{(i} P^{j)n}_{,n} \right) \\ &\quad + \text{source terms}, \end{aligned} \quad (25)$$

$$\begin{aligned} \dot{P}^{ij} &= \alpha_2 h^{l(i} \lambda^{j)l}_{,l} \\ &\quad + N^n P^{ij}_{,n} + Q\sqrt{h} \left(h^{mn} M^{ij}_{m,n} - 2h^{n(i} M^{j)k}_{k,n} \right) \\ &\quad + \text{source terms}. \end{aligned} \quad (26)$$

By construction, the “ λ -system” (20–26) is symmetric hyperbolic with respect to the inner product

$$\begin{aligned} \langle h^{ij}_1, P^{ij}_1, M^{ij}_1{}_k, \lambda_1, \lambda^i_1, \lambda^{ij}_1{}_k, \lambda^{ij}_1{}_{kl} \mid H^\lambda_E \mid h^{ij}_2, P^{ij}_2, M^{ij}_2{}_k, \lambda_2, \lambda^i_2, \lambda^{ij}_2{}_k, \lambda^{ij}_2{}_{kl} \rangle := \\ \int_{\Sigma_t} \left\{ e_{im} e_{jn} \bar{h}^{ij}_1 h^{mn}_2 + e_{im} e_{jn} \bar{P}^{ij}_1 P^{mn}_2 + e_{im} e_{jn} e^{kl} \bar{M}^{ij}_1{}_k M^{mn}_2{}_l + \bar{\lambda}_1 \lambda_2 \right. \\ \left. + e_{ij} \bar{\lambda}^i_1 \lambda^j_2 + e_{ip} e_{jq} e^{kr} \bar{\lambda}^{ij}_1{}_k \lambda^{pq}_2{}_r + e_{ip} e_{jq} e^{kr} e^{ls} \bar{\lambda}^{ij}_1{}_{kl} \lambda^{pq}_2{}_{rs} \right\} d\Sigma. \end{aligned} \quad (27)$$

The initial data for this purely dynamical set of equations consists of arbitrary functions

$$(h_0^{ij}, P_0^{ij}, M_0^{ij}{}_k, \lambda_0, \lambda_0^i, \lambda_0^{ij}{}_k, \lambda_0^{ij}{}_{kl}). \quad (28)$$

However, the dynamical degrees of freedom are extended by 40 λ -variables.

Clearly, for an arbitrary solution to Einstein's equations, $(h_E^{ij}, P_E^{ij}, M_E^{ij}{}_k)$, the embedded field configuration $(h^{ij}, P^{ij}, M^{ij}{}_k, \lambda, \lambda^i, \lambda^{ij}{}_k, \lambda^{ij}{}_{kl}) := (h_E^{ij}, P_E^{ij}, M_E^{ij}{}_k, 0, 0, 0, 0)$ is a solution to the λ -system. Conversely, every solution to the λ -system with vanishing λ -variables is also a solution to Einstein's equations. Due to this property, and since the solutions to the λ -system are unique, the λ -system naturally reproduces the dynamics on the constraint submanifold of general relativity.

Note that if the constraints are initially not satisfied, then, even when the λ -variables initially vanish, the λ -variables would pick up a non-zero value during time evolution. Hence, solutions to the λ -system corresponding to such initial data sets would not represent solutions to the complete set of Einstein's equations. In fact, they would not even solve the evolution equations of general relativity. However, for constraint- and λ -variables which initially are sufficiently close to zero, we suspect that the solutions asymptotically approach solutions to the Einstein equations. In the following section, we give analytical evidence that this conjecture could be true.

The system is by no means uniquely "extended", since one could still add non-principal (undifferentiated) terms, as long as they vanish when $\lambda = \lambda^i = \lambda^{ij}{}_k = \lambda^{ij}{}_{kl} = 0$. Such terms might be useful in order to treat the strongly non-linear regime. Of particular interest might be to choose the coefficients α_i and β_i , which control the damping in the λ -equations, to be quadratic functions of the basic variables $(h^{ij}, P^{ij}, M^{ij}{}_k)$, so that the damping becomes stronger at points where the non-linearities intensify.

It is fairly easy to implement similar schemes for alternative symmetric hyperbolic systems for the Einstein equations, as well as for symmetric hyperbolic systems for other theories with constraints, like, for instance, Yang-Mills theories. The strategy is the same: One writes equations with damping for first integrals of the constraints and modifies the evolution equations such that the extended system becomes symmetric hyperbolic. This can always be achieved, because the inclusion of the new equations modifies an off diagonal sector of the principal symbol matrix.

3 Asymptotic stability of the constraint propagation

The inclusion of the λ -terms into (1–3) affects, in turn, the evolution of the constraint quantities $\mathcal{C}, \mathcal{C}^i, \mathcal{C}^{ij}_k$, and \mathcal{C}^{ij}_{kl} . Recalculating the time derivative of these, and using (24–26), yields the constraint evolution equations for the new system,

$$\dot{\mathcal{C}} = N^n \mathcal{C}_{,n} + 3Q\sqrt{h}\mathcal{C}^k_{,k} - 2\alpha_4 h^{mn} \lambda^{kl}_{km,nl} + 2\alpha_0 h^{mn} \lambda_{,mn} + \dots, \quad (29)$$

$$\dot{\mathcal{C}}^i = N^n \mathcal{C}^i_{,n} + Q\sqrt{h} \left(h^{ik} \mathcal{C}_{,k} + h^{rs} \mathcal{C}^{ik}_{rk,s} \right) + \alpha_1 h^{m(n} \lambda^{i)}_{,mn} + \dots, \quad (30)$$

$$\begin{aligned} \dot{\mathcal{C}}^{ij}_k &= N^n \mathcal{C}^{ij}_{k,n} - 2Q\sqrt{h} \left(2\delta_k^{(i} \mathcal{C}^{j)} - h^{ij} h_{kl} \mathcal{C}^l \right) \\ &\quad + 2\alpha_3 h^{mn} \lambda^{ij}_{m,nk} + 2\alpha_4 h^{mn} \left(2\lambda^{ij}_{km,n} - h^{ij} h_{rs} \lambda^{rs}_{km,n} \right) \\ &\quad - \alpha_0 \left(2\delta_k^{(i} h^{j)l} \lambda_{,l} - h^{ij} \lambda_{,k} \right) + \dots, \end{aligned} \quad (31)$$

$$\begin{aligned} \dot{\mathcal{C}}^{ij}_{kl} &= N^n \mathcal{C}^{ij}_{kl,n} - 2Q\sqrt{h} \left(\delta_k^{(i} \mathcal{C}^{j)}_{,l} - \delta_l^{(i} \mathcal{C}^{j)}_{,k} \right) \\ &\quad + 2\alpha_4 \left(h^{mn} \lambda^{ij}_{km,nl} - h^{mn} \lambda^{ij}_{lm,nk} \right) \\ &\quad - \alpha_0 \left(\delta_k^{(i} h^{j)m} \lambda_{,ml} + \delta_l^{(i} h^{j)m} \lambda_{,mk} \right) + \dots. \end{aligned} \quad (32)$$

Again “...” represent undifferentiated terms that are linear in the constraint quantities and at least linear in (P^{ij}, M^{ij}_k) .

The propagation of the constraints is ruled by the system of equations consisting of (16–19) and (29–32), which determines whether or not the constraints asymptotically “decay” to zero. The crucial feature of this system is that the right hand side also contains non-principal terms. Roughly speaking, the operator on the right-hand side amplifies constraint violations if the matrix representing its action on periodic functions has any eigenvalue with a positive real part. On the other hand, if all the eigenvalues have a negative real part, the operator induces an asymptotic decay of these violations.

Instead of attacking the full non-linear problem as stated, which represents a problem well beyond the scope of present analytical techniques, we consider the linear regime of general relativity. That is, we restrict attention to 3-metrics of the form $h^{ij} = e^{ij} + \epsilon \gamma^{ij}$ with $e^{ij} = \delta^{ij}$ and $\epsilon \ll 1$. This implies that the variables (P^{ij}, M^{ij}_k) are of first-order in ϵ , as are the constraint quantities $(\mathcal{C}, \mathcal{C}^i, \mathcal{C}^{ij}_k, \mathcal{C}^{ij}_{kl})$ and the variables $(\lambda, \lambda^i, \lambda^{ij}_k, \lambda^{ij}_{kl})$. Thus, the terms represented by “...” in the equations (29–32) are of second order

in ϵ and shall be neglected. Without loss of generality, we restrict the following arguments to the case where the gauge source functions Q and N^i are constant. All arguments that follow refer to this linearized regime.

Although we lack a proof for the non-linear case, the following considerations provide analytical evidence for the asymptotic stability of the constraint propagation, in particular since the full evolution equations are quasi-linear.

For, as we believe, purely technical reasons, we adopt the following choice of coefficients: $\beta_0 = \beta_1 = \beta_3 = \beta_4 := \beta > 0$ and $\alpha_4 = \frac{\sqrt{3}}{2}\alpha_0$.

Theorem 1 *With the above assumptions, the constraint submanifold of the linearized Einstein equations is an asymptotically stable submanifold for the solutions to the linearized, λ -extended Einstein equations.*

We partition the proof of this theorem in several lemmas: We first show that the initial value problem is well posed and that the solutions stay bounded with time. Thus, it is possible to apply Laplace transformation techniques, which reduce the problem to the study of the eigenfrequencies of the system. For these frequencies, we show that the real part is non-positive, only approaches zero as the wave number goes to zero, and does so quadratically. Then stability follows from estimates in [KKL98].

Without loss of generality, we expand the linearized dynamical fields in Fourier integrals of the the following form:

$$\lambda(x, t) = \int \hat{\lambda}(k, t) \exp(ik \cdot x) d^3k , \quad (33)$$

$$\lambda^i(x, t) = \int \hat{\lambda}^i(k, t) \exp(ik \cdot x) d^3k , \quad (34)$$

$$\vdots \quad (35)$$

$$C^{ij}_k(x, t) = \int \hat{C}^{ij}_k(k, t) \exp(ik \cdot x) d^3k , \quad (36)$$

$$C^{ij}_{kl}(x, t) = \int \hat{C}^{ij}_{kl}(k, t) \exp(ik \cdot x) d^3k , \quad (37)$$

where $k \cdot x := k_i x^i$.

In terms of the Fourier transformed variables, equation (29–32) and (16–19) reduce to the system of ordinary differential equations given by

$$\dot{\hat{\lambda}} = -\beta \hat{\lambda} + \alpha_0 \hat{\mathcal{C}} , \quad (38)$$

$$\dot{\hat{\lambda}}^i = -\beta \hat{\lambda}^i + \alpha_1 \hat{\mathcal{C}}^i , \quad (39)$$

$$\dot{\hat{\lambda}}^{ij}_{kl} = -\beta \hat{\lambda}^{ij}_{kl} + \frac{\sqrt{3}}{2} \alpha_0 \hat{\mathcal{C}}^{ij}_{kl} , \quad (40)$$

$$\dot{\hat{\mathcal{C}}} = ik_n N^n \hat{\mathcal{C}} + 3iQ k_m \hat{\mathcal{C}}^m + \sqrt{3} \alpha_0 \hat{\lambda}^{rl}_{rm} k^m k_l - 2\alpha_0 \hat{\lambda} k^n k_n , \quad (41)$$

$$\dot{\hat{\mathcal{C}}}^i = ik_n N^n \hat{\mathcal{C}}^i + iQ \left(k^i \hat{\mathcal{C}} + k^r \hat{\mathcal{C}}^{in}_{rn} \right) - \frac{1}{2} \alpha_1 (k^n k_n \hat{\lambda}^i + k^i k_n \hat{\lambda}^n) , \quad (42)$$

$$\begin{aligned} \dot{\hat{\mathcal{C}}}^{ij}_{kl} &= ik_n N^n \hat{\mathcal{C}}^{ij}_{kl} - 2iQ \left(\delta_k^{(i} \hat{\mathcal{C}}^{j)} k_l - \delta_l^{(i} \hat{\mathcal{C}}^{j)} k_k \right) \\ &\quad - \sqrt{3} \alpha_0 \left(\hat{\lambda}^{ij}_{kr} k^r k_l - \hat{\lambda}^{ij}_{lr} k^r k_k \right) \\ &\quad + \alpha_0 \hat{\lambda} \left(\delta_k^{(i} k^{j)} k_l - \delta_l^{(i} k^{j)} k_k \right) , \end{aligned} \quad (43)$$

and

$$\dot{\hat{\lambda}}^{ij}_k = -\beta \hat{\lambda}^{ij}_k + \alpha_3 \hat{\mathcal{C}}^{ij}_k , \quad (44)$$

$$\dot{\hat{\mathcal{C}}}^{ij}_k = ik_n N^n \hat{\mathcal{C}}^{ij}_k - \alpha_3 \hat{\lambda}^{ij}_m k^m k_k + \hat{S}^{ij}_k , \quad (45)$$

where

$$\begin{aligned} \hat{S}^{ij}_k &:= -2Q \left(2\delta_k^{(i} \hat{\mathcal{C}}^{j)} - \delta^{ij} \hat{\mathcal{C}}_k \right) - \alpha_0 \left(2\delta^{(i}_k k^{j)} - h^{ij} k_k \right) \hat{\lambda} \\ &\quad + \sqrt{3} \alpha_0 k^m \left(2\hat{\lambda}^{ij}_{km} - h^{ij} h_{rs} \hat{\lambda}^{rs}_{km} \right) . \end{aligned} \quad (46)$$

This system of equations naturally splits up in two subsystems, since the equations (38–42) couple to the equations (44–45) only via the “source” term in (45). In the following, we will first establish that the solutions to the subsystem (38–42), and hence the coupling term in (45), asymptotically decay to zero. In a second step, we consider this coupling as a given, decaying source, and discuss the asymptotic behaviour of solutions to the subsystem (44–45).

Lemma 1 *Let \mathcal{H} be the space of the Fourier transformed $\hat{\lambda}^{ij}_{kl} \in L^2$, and let $\mathcal{D} \subset \mathcal{H}$ be the subspace defined by $\hat{\lambda}^{ij}_{ks} k^s = 0$. Then \mathcal{D} is invariant under time evolution, and the trivial solution $\hat{\lambda}^{ij}_{kl} = 0$ is asymptotically stable for the evolution restricted to \mathcal{D} .*

Proof: Multiplying equation (40) by k_m , antisymmetrizing, and using that $\hat{\mathcal{C}}^{ij}_{[kl]k_m} = \hat{M}^{ij}_{[k} k_l k_m] = 0$, we obtain

$$\dot{\hat{\lambda}}^{ij}_{[kl]k_m} = -\beta \hat{\lambda}^{ij}_{[kl]k_m} . \quad (47)$$

Next we note that for a function $\hat{\lambda}^{ij}_{kl}$ in \mathcal{H} , the component $(\hat{\lambda}^{ij}_{kl})^\parallel$ in \mathcal{D} is given by $(\hat{\lambda}^{ij}_{kl})^\parallel = \hat{\alpha}^{ij} k^r \varepsilon_{rkl}$, where $\hat{\alpha}^{ij} = \hat{\lambda}^{ij}_{[kl} k_{r]} \varepsilon^{klr} / (6k^2)$. Equation (47) is, therefore, equivalent to

$$(\dot{\hat{\lambda}}^{ij}_{kl})^\parallel = -\beta (\hat{\lambda}^{ij}_{kl})^\parallel , \quad (48)$$

which proves lemma 1.

By direct inspection of the evolution equations, it follows that the equation for the component of $\hat{\lambda}^{ij}_{kl}$ in the subspace \mathcal{D} decouples. It is, therefore, sufficient to concentrate on the evolution in the space $CF_\lambda \oplus CF_C$ which comprises those functions $(\hat{\lambda}, \hat{\lambda}^i, \hat{\lambda}^{ij}_{kl}, \hat{C}, \hat{C}^i, \hat{C}^{ij}_{kl}) \in L^2$ for which $\hat{\lambda}^{ij}_{kl} \in \mathcal{D}^\perp$. Here, \mathcal{D}^\perp denotes the L^2 complement of \mathcal{D} in \mathcal{H} , which, as easily seen, is spanned by the elements $\hat{\lambda}^{ij}_{kl} \in L^2$ satisfying $\hat{\lambda}^{ij}_{[kl} k_{m]} = 0$.⁵ Since for the constraint variable \hat{C}^{ij}_{kl} , the same property is fulfilled, $\hat{C}^{ij}_{[kl} k_{m]} = 0$, this shows that the spaces CF_λ and CF_C are naturally isomorphic, $CF_\lambda \approx CF_C =: CF$.

To simplify the notation, and to display the structure of the evolution equations considered more transparently, let us introduce the following operator \mathbf{E} acting on functions $\mathbf{v} := (v, v^i, v^{ij}_{kl})$ in CF :

$$\mathbf{E}(\mathbf{v}) := \left(E(\mathbf{v}), E^i(\mathbf{v}), E^{ij}_{kl}(\mathbf{v}) \right) , \quad (49)$$

where

$$E(\mathbf{v}) := \sqrt{3}\alpha_0 v^{rl} {}_{rm} k^m k_l - 2\alpha_0 v k^n k_n , \quad (50)$$

$$E^i(\mathbf{v}) := -\frac{1}{2}\alpha_1 \left(v^i k^n k_n + v^n k^i k_n \right) , \quad (51)$$

$$\begin{aligned} E^{ij}_{kl}(\mathbf{v}) &:= -\sqrt{3}\alpha_0 \left(v^{ij} {}_{kn} k^n k_l - v^{ij} {}_{ln} k^n k_k \right) \\ &\quad + \alpha_0 v \left(\delta_k^{(i} k^{j)} k_l - \delta_l^{(i} k^{j)} k_k \right) . \end{aligned} \quad (52)$$

Taking advantage of these definitions, the evolution system (38-43) restricted to the subspace $CF \oplus CF$ can be rewritten as

$$\frac{d}{dt} \begin{pmatrix} \boldsymbol{\lambda} \\ \mathbf{C} \end{pmatrix} = \begin{pmatrix} -\mathbf{S} & \boldsymbol{\Gamma} \\ \mathbf{E} & i\mathbf{A} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \mathbf{C} \end{pmatrix} =: \mathbf{P} \begin{pmatrix} \boldsymbol{\lambda} \\ \mathbf{C} \end{pmatrix} , \quad (53)$$

⁵For functions $\hat{\lambda}^{ij}_{kl}$ in \mathcal{D} only the components along k^m are non-trivial, $\hat{\lambda}^{ij}_{kl} = -2k_{[l} \hat{\lambda}^{ij} {}_{k]m} k^m / k^2$. This can be seen by solving $\hat{\lambda}^{ij}_{[kl} k_{m]} = 0$, and by using the anti-symmetry in the lower indices of $\hat{\lambda}^{ij}_{kl}$.

where \mathbf{S} and $\mathbf{\Gamma}$ are diagonal matrices determined by the parameters β and α_i , respectively, and where \mathbf{A} is an operator of the form $\mathbf{A}^m k_m$.

In a next step, we show that the operator $e^{\mathbf{P}t}$ is bounded with respect to a suitably chosen norm. To this end, we first establish the following

Lemma 2 *The operator $H_\lambda := -\mathbf{\Gamma}^{-1}H_c\mathbf{E}$ considered as a matrix-valued field on the Fourier space \mathbf{R}^3 is symmetric and coercive with respect to the inner product $\langle \mathbf{u}, \mathbf{v} \rangle := \bar{u}v + e_{ij}\bar{u}^i v^j + e_{ip}e_{jq}e^{kr}e^{ls}\bar{u}^{ij}_{kl}v^{pq}_{rs}$. That is, $\langle \mathbf{u}, H_\lambda \mathbf{v} \rangle = \langle H_\lambda \mathbf{u}, \mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v} \in CF$, and there exists a constant $c > 0$ such that $\langle \mathbf{u}, H_\lambda \mathbf{u} \rangle \geq ck^2 \langle \mathbf{u}, \mathbf{u} \rangle$ for all $\mathbf{u} \in CF$.*

Proof: We have

$$\begin{aligned}
& \langle \mathbf{u}, \mathbf{\Gamma}^{-1}H_c \mathbf{E}(\mathbf{v}) \rangle - \langle \mathbf{\Gamma}^{-1}H_c \mathbf{E}(\mathbf{u}), \mathbf{v} \rangle \\
&= \frac{1}{3\alpha_0}\bar{u} \left(-2\alpha_0 v k_n k^n + \sqrt{3}\alpha_0 v^{kl}_{km} k^m k_l \right) \\
&\quad + \frac{1}{\alpha_1}\bar{u}^i \left(-\frac{\alpha_1}{2} \left(v^j k_n k^n + k^j v^l k_l \right) \right) e_{ij} \\
&\quad + \frac{1}{2\sqrt{3}\alpha_0} e_{im} e_{jn} e^{kp} e^{lq} \bar{u}^{ij}_{kl} \left(-2\sqrt{3}\alpha_0 v^{mn}_{ps} k^s k_q + 2\alpha_0 v \delta^m_p k^n k_q \right) \\
&\quad - \frac{1}{3\alpha_0} v \left(-2\alpha_0 \bar{u} k_n k^n + \sqrt{3}\alpha_0 \bar{u}^{kl}_{km} k^m k_l \right) \\
&\quad - \frac{1}{\alpha_1} v^i \left(-\frac{\alpha_1}{2} \left(\bar{u}^j k_n k^n + k^j \bar{u}^l k_l \right) \right) e_{ij} \\
&\quad + \frac{1}{2\sqrt{3}\alpha_0} e_{im} e_{jn} e^{kp} e^{lq} v^{ij}_{kl} \left(-2\sqrt{3}\alpha_0 \bar{u}^{mn}_{ps} k^s k_q + 2\alpha_0 \bar{u} \delta^m_p k^n k_q \right) \\
&= 0.
\end{aligned}$$

The remaining part of the proof is given in appendix A, where we show that H_λ is coercitive with constant $c = 1/4$.

With the help of lemma 2, it is now easy to prove

Lemma 3 *The matrix-valued fields \mathbf{P}_\pm ,*

$$\mathbf{P}_+ := \begin{pmatrix} \mathbf{S} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{P}_- := \begin{pmatrix} 0 & \mathbf{\Gamma} \\ \mathbf{E} & i\mathbf{A} \end{pmatrix}, \quad (54)$$

are hermitian respectively anti-hermitian with respect to the inner product

$$\langle (\boldsymbol{\lambda}_1, \mathbf{C}_1), H_T(\boldsymbol{\lambda}_2, \mathbf{C}_2) \rangle := \langle \boldsymbol{\lambda}_1, H_\lambda \boldsymbol{\lambda}_2 \rangle + \langle \mathbf{C}_1, H_c \mathbf{C}_2 \rangle. \quad (55)$$

Proof: Since $\mathbf{S} = \beta \mathbf{I}$, the statement for \mathbf{P}_+ is trivially true. The anti-symmetry of \mathbf{P}_- follows directly from lemma 2, and the symmetry of \mathbf{A} with respect to H_c .

Taking advantage of lemma 3, we now obtain the following important estimate for the operator $\mathbf{P} = \mathbf{P}_+ + \mathbf{P}_-$:

$$H_T \mathbf{P} + \mathbf{P}^\dagger H_T = H_\lambda \mathbf{S} + \mathbf{S} H_\lambda = -2\beta H_\lambda \leq -2\beta \frac{1}{4} k_n k^n \leq 0, \quad (56)$$

where, for any Hermitian matrix \mathbf{M} , the inequality $\mathbf{M} \leq 0$ means $\langle \mathbf{v}, \mathbf{M} \mathbf{v} \rangle \leq 0$ for all \mathbf{v} .⁶

The symmetry and coercivity of the operator H_λ imply that H_λ can be used to define a scalar product on a (dense) subspace $\mathcal{D}(CF)$ of the Hilbert space CF ,⁷ which, in turn, shows that the operator $H_T = H_c + H_\lambda$ gives rise to a scalar product on $CF \oplus \mathcal{D}(CF)$.

As is well known (see, for instance, [KL89]), the estimate (56) implies that for all $t > 0$, the operator $e^{\mathbf{P}t}$ is bounded with respect to the norm defined by H_T . Hence, the initial value problem for the system considered is well posed. Moreover, all solutions with initial data which are bounded with respect to this norm remain bounded for all positive times. Thus Laplace transformation techniques can be applied [KL89], and the relevant questions are the *sign* of the real part of the eigenvalues of \mathbf{P} , and how fast they approach zero as the wave number $k = \sqrt{k^i k_i}$ goes to zero. Hence, the proof is reduced to the eigenvalue problem for the operator \mathbf{P} ,

$$\mathbf{P} \begin{pmatrix} \lambda_s \\ \mathbf{C}_s \end{pmatrix} = s \begin{pmatrix} \lambda_s \\ \mathbf{C}_s \end{pmatrix}. \quad (57)$$

Then we have the following

Lemma 4 *The eigenvalues of the above system have non-positive real part and furthermore there exist positive constants c_1 and w_1 such that*

$$\Re(s) \leq -c_1 \frac{k^2}{w_1 + k^2} \quad (58)$$

for all wave vectors k_i

⁶Clearly, there are other possible choices of the operators \mathbf{S} which lead to the same inequality. Here we have restricted to the simplest possibility, but for practical applications, alternative choices might be better suited.

⁷In physical space, the relevant function space equipped with the norm corresponding to the above scalar product is very similar to the Sobolev space H_0^1 .

Proof: From the λ -rows of the eigenvalue equation, we get

$$\mathbf{C}_s = (s + \beta)\mathbf{\Gamma}^{-1}\lambda_s. \quad (59)$$

Using this in the \mathbf{C} -rows, we next obtain

$$\left(\mathbf{E} + (s + \beta)(-s\mathbf{I} + i\mathbf{A})\mathbf{\Gamma}^{-1}\right)\lambda_s = 0. \quad (60)$$

Multiplying the above equation by $-(\mathbf{\Gamma}^{-1})^\dagger \mathbf{H}_c$ from the left and subsequently contracting with λ_s , we find the following second order equation for the eigenvalue s :

$$\langle \lambda_s, H_\lambda \lambda_s \rangle + (s + \beta) \left(s \langle \lambda_s, (\mathbf{\Gamma}^{-1})^\dagger H_c \mathbf{\Gamma}^{-1} \lambda_s \rangle - i \langle \lambda_s, (\mathbf{\Gamma}^{-1})^\dagger H_c \mathbf{A} \mathbf{\Gamma}^{-1} \lambda_s \rangle \right) = 0. \quad (61)$$

The established properties of the involved operators imply that

$$c(k_i^0)k^2 := \frac{\langle \lambda_s, H_\lambda \lambda_s \rangle}{\langle \lambda_s, (\mathbf{\Gamma}^{-1})^\dagger H_c \mathbf{\Gamma}^{-1} \lambda_s \rangle} \quad (62)$$

is positive for $k_i \neq 0$, and that

$$b(k_i^0)k := \frac{\langle \lambda_s, (\mathbf{\Gamma}^{-1})^\dagger H_c \mathbf{A} \mathbf{\Gamma}^{-1} \lambda_s \rangle}{\langle \lambda_s, (\mathbf{\Gamma}^{-1})^\dagger H_c \mathbf{\Gamma}^{-1} \lambda_s \rangle} \quad (63)$$

is real, where k_i^0 denotes the unit vector in the direction of k_i , and k is the norm of k_i . Thus we have for each direction of k^i

$$(s + \beta)(s - ibk) + ck^2 = 0 \quad (64)$$

with β, b, c real and β, c positive. For this equation we prove in appendix B that the real part of the roots satisfies the desired inequality, which establishes the result for each direction of the wave vector k_i . Using the maximal values of $-c_1 \frac{k^2}{w_1 + k^2}$ on the 2-sphere of directions of k_i , we obtain the final inequality.

With this bound on the decay constants, it is now easy to prove asymptotic stability for the subsystem (38–43). Splitting the set of solutions in a part with frequencies with $k < 1$, and another with $k \geq 1$, the above bound tells us that the solutions of the higher frequency part decay faster than $e^{-\frac{c_1 t}{w_1 + 1}}$, while the decay of the solutions of the low frequency part can be estimated as in [KKL98, lemma 1 and 2 of section III].

We now turn attention to the second set of equations, given by (44) and (45), and establish the following

Lemma 5 *Let \mathcal{H}_3 be the space of the Fourier transformed $(\hat{\lambda}^{ij}_k, \hat{C}^{ij}_k) \in L^2$. Then \mathcal{H}_3 is invariant under time evolution, and the trivial solution $(\hat{\lambda}^{ij}_k, \hat{C}^{ij}_k) = 0$ is asymptotically stable for the evolution restricted to \mathcal{H}_3 .*

Proof: In a first step, we discuss the equation for the component of a solution in the subspace

$$\mathcal{D}_3 := \{ (\hat{\lambda}^{ij}_k, \hat{C}^{ij}_k) \in L^2 \mid \hat{\lambda}^{ij}_m k^m = \hat{C}^{ij}_m k^m = 0 \} . \quad (65)$$

Taking advantage of equation (8), (12), and (44), we obtain

$$\dot{\hat{\lambda}}^{ij}_{[k} k_{l]} = -\beta \hat{\lambda}^{ij}_{[k} k_{l]} + \alpha_3 \hat{C}^{ij}_{[k} k_{l]} , \quad (66)$$

$$i \hat{C}^{ij}_{[k} k_{l]} = \hat{C}^{ij}_{kl} - \frac{1}{2} \delta^{ij} \delta_{mn} \hat{C}^{mn}_{kl} , \quad (67)$$

which implies that the space \mathcal{D}_3 is invariant under time evolution. As already shown, the constraint variable \hat{C}^{ij}_{kl} asymptotically decays to zero. The dynamics in \mathcal{D}_3 is, therefore, described by a system of ordinary differential equations of the form $\dot{u} = -u + f$, where f is a given source with $f \rightarrow 0$ as $t \rightarrow \infty$. Since any solution to this system satisfies $u \rightarrow 0$ as $t \rightarrow \infty$,⁸ it follows that solutions in \mathcal{D}_3 decay with time.

It remains to discuss the complementary subspace \mathcal{D}_3^\perp ,

$$\mathcal{D}_3^\perp = \{ (\hat{\lambda}^{ij}_k, \hat{C}^{ij}_k) \in L^2 \mid \hat{\lambda}^{ij}_{[k} k_{l]} = \hat{C}^{ij}_{[k} k_{l]} = 0 \} . \quad (68)$$

For the component of a solution in this subspace, we find

$$\frac{d}{dt} \begin{pmatrix} \boldsymbol{\lambda}_3 \\ \mathbf{C}_3 \end{pmatrix} = \begin{pmatrix} -\beta & \alpha_3 \\ -\alpha_3 k^2 & i k_m N^m \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda}_3 \\ \mathbf{C}_3 \end{pmatrix} - \begin{pmatrix} 0 \\ F^{ij} k_k \end{pmatrix} , \quad (69)$$

where $(\boldsymbol{\lambda}_3, \mathbf{C}_3) := (\hat{\lambda}^{ij}_k, \hat{C}^{ij}_k)^\perp \in \mathcal{D}_3^\perp$, and where $\hat{F}^{ij} k_k$ is a shorthand for the perpendicular component of the source term \hat{S}^{ij}_k , $\hat{F}^{ij} = \hat{S}^{ij}_m k^m / k^2$. Thus, as expected, the subspace \mathcal{D}_3 is invariant as well.

Since \hat{P}^{ij} and consequently $\hat{C}^i/|k| = \hat{P}^{im} k_m / |k|$ are contained in L^2 , equation (46) implies that the same is true for \hat{F}^{ij} , $\hat{F}^{ij} \in L^2$. Furthermore,

⁸For a proof, choose T such that $f(t) < \varepsilon/2$ for all $t > T$. Since the general solution to the above system is given by $u(t) = e^{-t}(u(0) + \int_0^t e^{\tilde{t}} f(\tilde{t}) d\tilde{t})$, it follows that $u(t) \leq e^{-t}(u(0) + \int_0^T e^{\tilde{t}} f(\tilde{t}) d\tilde{t} - \varepsilon e^T/2) + \varepsilon/2$. Hence, for a sufficiently large time $t_0 > T$, the absolute value of the first term becomes smaller than $\varepsilon/2$, which implies $|u(t)| < \varepsilon$ for all $t > t_0$.

the real part of the eigenvalues of the system (69) can, as in lemma 4, be estimated by the inequality (58), albeit for different constants. Adopting a similar reasoning as in the previous discussion, and applying lemma 1 and 2 of [KKL98] to this system, it follows that solutions in \mathcal{D}_3^\perp also decay with time.

This completes the proof of lemma 5 and hence the proof of our main result.

4 Conclusions

In the present paper we have shown that an arbitrary system of symmetric hyperbolic evolution equations with constraints admits extensions to symmetric hyperbolic systems which reproduce the original dynamics on the embedded constraint submanifold. We have given analytical evidence that the class of extensions proposed is sufficiently rich to contain systems for which the embedded constraint submanifold is an attractor of the time evolution. For the Einstein equations, we have constructed an extended evolution system for which, at least in the linearized case, this property is fulfilled.

It is natural to expect that, by making use of techniques developed in [KKL98], the results proven for the linearized Einstein equations can be generalized to the regime of non-linear general relativity describing space-times in the vicinity of Minkowski space. However, to establish similar results for more extended regions of the phase space of general relativity is well beyond the scope of present analytic techniques.

Numerical experiences with the Navier-Stokes equations for incompressible fluids show that asymptotic stability of the constraint submanifold is essential for accurate results [Kre]. For this system, techniques with a very similar effect have been used to include the incompressibility constraint into the evolution equations. On the basis of this observation, and the results established for linearized gravity, we suspect that the extensions of Einstein's equations constructed could be of interest when obtaining numerical solutions to general relativity. Numerical experiments testing aspects of this conjecture are in progress.

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A Proof of coercivity

In this appendix we show that

$$\langle \mathbf{u}, H_\lambda \mathbf{u} \rangle \geq \frac{1}{4} k_n k^n \quad (70)$$

for unitary \mathbf{u} satisfying $u^{ij}_{rs} = u^{(ij)}_{[rs]}$ and $u^{ij}_{[rs]k_l} = 0$, as needed for lemma 2. We treat this as the problem of extremizing the quadratic function of \mathbf{u} on the left-hand side of (70) under the constraint condition $\langle \mathbf{u}, \mathbf{u} \rangle = 1$.

From (50–52), we obtain

$$\begin{aligned} \langle \mathbf{u}, H_\lambda \mathbf{u} \rangle + \tau k^n k_n (1 - \langle \mathbf{u}, \mathbf{u} \rangle) = \\ \frac{2}{3} k^n k_n u \bar{u} + \frac{1}{2} k^n k_n u^i \bar{u}_i + \frac{1}{2} u^i k_i \bar{u}_n k^n + u^{ij}_{rs} k^s \bar{u}_{ij}{}^{rm} k_m - \frac{\sqrt{3}}{3} \bar{u} u^{ij}_{is} k^s k_j \\ - \frac{1}{\sqrt{3}} u \bar{u}_{ij}{}^{is} k_s k^j + \tau k^n k_n \left(1 - u \bar{u} - u^i \bar{u}_i - u^{ij}_{rs} \bar{u}_{ij}{}^{rs} \right), \end{aligned} \quad (71)$$

where τ is a Lagrange multiplier and where indices are raised and lowered with e_{ij} . To simplify the algebra, we choose a basis in which $k^n = (0, 0, k)$. Then $u^{ij}_{rs} = 0$, except when $s = 3$. Hence,

$$\begin{aligned} F(\mathbf{u}, \tau) := \langle \mathbf{u}, H_\lambda \mathbf{u} \rangle + \tau k^n k_n (1 - \langle \mathbf{u}, \mathbf{u} \rangle) = \\ k^2 \left(\frac{2}{3} u \bar{u} + \frac{1}{2} u^i \bar{u}_i + \frac{1}{2} u^3 \bar{u}_3 + u^{ij}_{r3} \bar{u}_{ij}{}^{r3} - \frac{1}{\sqrt{3}} \bar{u} u^{i3}{}_{i3} \right. \\ \left. - \frac{1}{\sqrt{3}} u \bar{u}_{i3}{}^{i3} + \tau \left(1 - u \bar{u} - u^i \bar{u}_i - u^{ij}_{rs} \bar{u}_{ij}{}^{rs} \right) \right). \end{aligned} \quad (72)$$

The function $F(\mathbf{u}, \tau)$ is extremized at points (\mathbf{u}, τ) where

$$\frac{\partial F}{\partial \mathbf{u}} + \frac{\partial F}{\partial \bar{\mathbf{u}}} = 0, \quad (73)$$

$$\frac{\partial F}{\partial \mathbf{u}} - \frac{\partial F}{\partial \bar{\mathbf{u}}} = 0 , \quad (74)$$

$$\frac{\partial F}{\partial \tau} = 0 . \quad (75)$$

Equation (75) is the requirement that \mathbf{u} has unit length. Equation (73) and (74) constitute a homogeneous linear system of equations for the real and imaginary parts of \mathbf{u} . Since \mathbf{u} cannot vanish, the determinant of the linear system has to vanish. Up to numerical factors, this is given by

$$(2\tau - 1)^{13} (\tau - 1)^2 \left(\tau - \frac{1}{6} \right) . \quad (76)$$

As easily verified, $\tau = 1$ yields the following minimal value of $F(\mathbf{u}, \tau)$ (when evaluated at unit \mathbf{u} such that (73)–(74) are satisfied):

$$F(\mathbf{u}_{min}, 1) = \frac{1}{4} k^2 , \quad (77)$$

from which (70) follows. The other extreme values of $F(\mathbf{u}, \tau)$ are $(5/3)k^2$ and $(1/2)k^2$ for $\tau = 1/6, 1/2$.

B On the proof of lemma 4

In this appendix we prove that the roots s_{\pm} of the polynomial

$$P(s) = s^2 + s(\beta - ibk) + ck^2 \quad (78)$$

are subject to the inequality

$$\Re(s_{\pm}) \leq -c_1 \frac{k^2}{w_1 + k^2} , \quad (79)$$

where $c_1 = \beta c / (b^2 + 4c)$ and $w_1 = \beta^2 / (b^2 + 4c)$. As in the body of the text, it is assumed that the parameters of P are real, and that β and c are strictly positive.

To begin with, let us rewrite the polynomial P , and the above estimate in terms of suitably rescaled parameter. Defining

$$\tilde{s} = s/\beta , \quad \tilde{k} = 2k\sqrt{b^2 + 4c}/\beta , \quad \tilde{b} = b/\sqrt{b^2 + 4c} , \quad (80)$$

and dropping tildes, we obtain for the polynomial

$$P/\beta^2 = s^2 + s(1 - ibk) + (1 - b^2)k^2/4. \quad (81)$$

The estimate for the roots in terms of the scaled parameters assumes the form

$$\Re(s_{\pm}) \leq -\frac{\gamma^2}{4} \frac{k^2}{1 + k^2}, \quad (82)$$

where $\gamma^2 := 1 - b^2 \in (0, 1]$.

As easily verified, the roots of the scaled polynomial satisfy

$$\max\{\Re(2s_+), \Re(2s_-)\} = -1 + |\Re \sqrt{1 - k^2 + 2ibk}|. \quad (83)$$

It is, therefore, sufficient to show that

$$|\Re \sqrt{1 - k^2 + 2ibk}| \leq 1 - \frac{\gamma^2}{2} \frac{k^2}{1 + k^2}. \quad (84)$$

To give a proof of this inequality, we first evaluate the identity

$$2|\Re \sqrt{z}|^2 = |z| + \Re(z) \quad (85)$$

for $z := 1 - k^2 + 2ibk$,

$$\begin{aligned} 2|\Re \sqrt{z}|^2 &= \sqrt{(1 - k^2)^2 + 4b^2k^2} + (1 - k^2) \\ &= \sqrt{(1 + k^2)^2 - 4\gamma^2k^2} - (1 + k^2) + 2. \end{aligned}$$

Hence,

$$\begin{aligned} |\Re \sqrt{z}|^2 &= 1 + (1 + k^2) \left\{ \sqrt{1 - 4\gamma^2k^2/(1 + k^2)^2} - 1 \right\} / 2 \\ &\leq 1 - \gamma^2 \frac{k^2}{1 + k^2}, \end{aligned} \quad (86)$$

where we have used the estimate $\sqrt{1 - x} \leq 1 - x/2$, which holds for $x \leq 1$. Taking advantage of the latter estimate once again, it follows that

$$|\Re \sqrt{z}| \leq 1 - \frac{\gamma^2}{2} \frac{k^2}{1 + k^2}, \quad (87)$$

which completes the proof of our claim.

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